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Restrained 2-Domination in Graphs

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Abstract: *- Let G = (V, E) be a graph. A set $S \subseteq V(G)$ is a restrained 2- dominating set if every vertex of $V(G)\setminus S$ is adjacent to at least two vertices in S and every vertex of $V(G)\setminus S$ is adjacent to a vertex in $V(G)\setminus S$. The restrained 2- domination number of G, denoted by γ_{r2} (G), is the smallest cardinality of a restrained 2- dominating set of G. In this paper we study restrained 2- domination in graphs and obtain some results.*

Keywords: *- Domination, 2-domination, restrained domination*
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I. Introduction

Let G = (V, E) be a finite undirected graph without loops or multiple edges. The order and size of G are denoted by *p* and *q* respectively. For graph theoretical terms we refer to Harary [4] and for terms related to domination we refer to Haynes et al. [9]. For every vertex $v \in V$, the open neighborhood N(*v*) is the set {$u \in$ V(G)/*uv* $\in$ E(G)} and the closed neighborhood N[v] is the set N[v] = N(v) U {v}. A vertex in a graph G dominates itself and its neighbors. An end-vertex or a pendant vertex in a graph G is a vertex of degree one and a support vertex is one that is adjacent to an end-vertex.

A subset S of V in a graph G is said to be a dominating set of G if every vertex of $V(G)\setminus S$ is adjacent to at least one vertex in S. The domination number γ(G) is the minimum cardinality of a dominating set of G. A dominating set S with $|S| = \gamma$ is called a γ-set.

The concept of restrained domination was introduced by Telle and Proskurowski [5] indirectly as a vertex partitioning problem. Cyman and Raczek [6] introduced the concept of total restrained domination.

A dominating set $S \subseteq$ V(G) is a restrained dominating set if every vertex of $V(G)\setminus S$ is adjacent to a vertex in *S* and every vertex of $V(G)\setminus S$ is adjacent to a vertex in $V(G)\setminus S$. The restrained domination number of G denoted by γ_r (G) is the minimum cardinality of a restrained dominating set of G. A restrained dominating set S with $|S| = \gamma_r$ is called a γ_r-set.

The concept of 2-domination was introduced by Fink and Jacobson [1, 2].

A dominating set $S \subseteq$ V(G) is a 2-dominating set if every vertex of $V(G)\setminus S$ is adjacent to at least two vertices in S. The 2-domination number of G denoted by γ_2(G) is the minimum cardinality of a 2-dominating set of G. A restrained dominating set S with $|S| = \gamma_2$ is called a γ_2-set.

In this paper we study restrained 2-domination in graphs and obtain some results.

II. Main Results

A set $S \subseteq$ V(G) is a restrained 2-dominating set of G if every vertex of $V(G)\setminus S$ is adjacent to at least two vertices in S and to a vertex in $V(G)\setminus S$. The restrained 2-domination number of G denoted by γ_{r2} (G) is the minimum cardinality of a restrained 2-dominating set of *G*. A restrained 2- dominating set of cardinality $\gamma_{r2}(G)$ is called a $\gamma_{r2}(G)$-set.

Let S be any restrained 2-dominating set of a graph G. Then the degree of every vertex of $V(G)\setminus S$ is at least three and all the vertices of degree less than three must be included in S. Thus S contains all pendant vertices and its supporting vertices.

If p is the number of vertices in G then bounds for $\gamma_{r2}(G)$ can be stated as:

Lemma 2.1: For any graph G, $2 \le \gamma_{r2}(G) \le p$.

Lemma 2.2: For any graph G with $\delta(G) \ge 3, \gamma_{r2}(G) \le p-2$.

Since any restrained 2-dominating set is also a 2-dominating set, so the inequality $\gamma_2(G) \le \gamma_{r2}(G)$ for any graph *G* is obvious. On the other hand if a graph G has a $\gamma_2(G)$ set S such that G [$V(G) \setminus S$] has no isolated vertices, then S is also a restrained 2- dominating set, so $\gamma_{r2}(G) \le \gamma_2(G)$.

Hence we have the following characterization of those graphs for which the 2- restrained domination number is equal to the 2-domination number.

Theorem 2.3: For a graph G, $\gamma_2(G) = \gamma_{r2}(G)$ if and only if G has a $\gamma_2(G)$ -set S such that G [$V(G) \setminus S$] has no-isolated vertices.

Suppose that $n \ge 1$ and let $k \in \{1, 2, ...n-2, n-1, n\}$. Let G be the graph obtained from P_{n-k} , the path on $n-k$ vertices, by adding a set of vertices $\{u, v, v_1, v_2, ...v_{k-1}\}$ and joining the vertices *u* and *v* to each of the vertices in $V(P_{n-k}) \cup \{v_1, v_2, ...v_{k-1}\}$. Then $\gamma_2(G) = 2$ and $\gamma_{r2}(G) = k$. Hence we have the result-

Theorem 2.4: There exists a graph G for which $\gamma_{r2}(G) - \gamma_2(G)$ can be made arbitrarily large.

Let K_n, C_n and P_n denote, respectively, the complete graph, the cycle graph and the path graph of order n. Also, let W_n and F_n denote the wheel graph and fan graph of order n. $K_{m,n}$ is a complete bipartite graph on *m* and *n* vertices and $K_{1,\, n-1}$ a star graph of order n. A galaxy is a disjoint union of stars. A bistar B(r, s) is a graph obtained by joining the centres of two stars $K_{1,\, r}$ and $K_{1,\, s}$ by an edge.

If a vertex is of degree 2 in G then both of its adjacent vertices must be included in restricted 2-dominating set S.

Theorem 2.5:

(i) If n is a positive integer, then $\gamma_{r2}(K_n) = n \quad \text{for n} \le 3$
$\qquad 2 \quad \text{for n} > 3.$

(ii) If $n \ge 3$ is a positive integer, then $\gamma_{r2}(C_n) = n$.

(iii) If *n* is a positive integer, then $\gamma_{r2}(P_n) = n$.

(iv) If $n \ge 4$ is a positive integer, then $\gamma_{r2}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

(v) If $n \ge 4$ is a positive integer, then $\gamma_{r2}(F_n) = \left\lceil \frac{\text{n}}{2} \right\rceil$.

(vi) For any positive integers *m* and *n*, $\gamma_{r2}(K_{m,n}) = 4 \quad \text{for m} \ge 3 \text{ and n} \ge 3$
$\qquad = m+n \quad \text{otherwise.}$

(vii) If $n \ge 2$ is a positive integer, then $\gamma_{r2}(K_{1,n-1}) = n$.

(viii) For a Bistar, $\gamma_{r2}\left[B(r,s)\right] = r+s \quad \text{for r} \ge 2 \text{ and s} \ge 2$
$\qquad = r+s+2 \quad \text{otherwise.}$

Proof : (i) Domination number of complete graph is one and 2-domination number is 2. A vertex of K_n is adjacent to all the remaining vertices, so for n > 3 a minimal 2- dominating set is a minimal restrained 2-dominating set, hence the result.

ii, iii) For C_n and P_n as max degree of a vertex is 2, every vertex must be included in the restrained 2-dominating set. Hence the result.

iv) For W_n, n > 3, the center vertex must be in the 2-dominating set. From the cycle C_{n-1}, we take alternate set of the vertices in restrained 2-dominating set. Hence the result.

vi) For Km,n two vertices each from two parts is sufficient fot restrained 2-domination for m, n >= 3, otherwise we must include all the vertices. Similarly for star graph we must include all vertices into restrained 2-dominating set as except the central vertex, all vertices are of degree < 2. Also for bi-star similar argument follows.

Now we determine the restrained 2-domination number of the complements of some families of graphs mentioned above. Here $\bar{G}$ denotes the complement of a graph G.

Theorem 2.6:

(i) If n is a positive integer, then for complement of complete Graphs K_n $\gamma_{r2}(\bar{K}_n) = n.$

(ii) If $n \geq 3$ is a positive integer, then for complement of cycle graph C_n $\begin{aligned}\gamma_{r2}(\bar{C}_n) &= n \quad \text{for } n \leq 5 \\ &= 3 \quad \text{for } n > 5.\end{aligned}$.

(iii) If n is a positive integer, then for complement of path graph P_n $\begin{aligned}\gamma_{r2}(\bar{P}_n) &= n \quad \text{for } n \leq 4 \\ &= 4 \quad \text{for } n > 4.\end{aligned}$

(iv) If $n \geq 3$ is a positive integer, then for complement of wheel graph W_n $\begin{aligned}\gamma_{r2}(\bar{W}_n) &= n \quad \text{for } n \leq 6 \\ &= 4 \quad \text{for } n > 6.\end{aligned}$

(v) If $n \geq 3$ is a positive integer, then for complement of fan graph F_n $\begin{aligned}\gamma_{r2}(\bar{F}_n) &= n \quad \text{for } n \leq 5 \\ &= 4 \quad \text{for } n > 5.\end{aligned}$

(vi) If m, n are positive integers, then for complement of complete bipartite graph

$$\begin{aligned}\gamma_{r2}(\bar{K}_{m,n}) &= m+n \qquad \text{for } m \leq 3 \text{ and } n \leq 3 \\ &= 4 \qquad \text{for } m \geq 4 \text{ and } n \geq 4 \\ &= 5 \qquad \text{otherwise.}\end{aligned}$$

(vii) If n is a positive integer, then for complement of star graph $K_{1,n}$ $\begin{aligned}\gamma_{r2}(\bar{K}_{1,n-1}) &= 4 \quad \text{for } n = 4 \\ &= 3 \quad \text{otherwise.}\end{aligned}$

(viii) For the complement of Petersen graph G, $\gamma_{r2}(\bar{G}) = 4$.

(ix) For complement of Bistar B(r, s), $\begin{aligned}\gamma_{r2}\left[\bar{B}(r,s)\right] &= r+s+1 \quad \text{for } (r,s) \neq (0,0), (1 \\ &= 3 \qquad \text{otherwise.}\end{aligned}$ From the above results we can get the following equalities-

Remark 2.7:

(i) If $n \geq 4$ is a positive integer, then $\gamma_{r2}(\bar{W}_n) = 1 + \gamma_{r2}(\bar{C}_{n-1})$.

(ii) If $n \geq 4$ is a positive integer, then $\gamma_{r2}(\bar{F}_n) = 1 + \gamma_{r2}(\bar{P}_{n-1})$.

(iii) For any positive integers m and n, $\gamma_{r2}(\bar{K}_{m,n}) = \gamma_{r2}(K_m) + \gamma_{r2}(K_n)$.

(iv) If $n \geq 2$ is a positive integer, then $\gamma_{r2}(\bar{K}_{1,n-1}) = 1 + \gamma_{r2}(K_{n-1})$.

We can prove the inequality for the cross product of two graphs G and H if the two graphs posses $\gamma_{r2}(G)$-set and a $\gamma_{r2}(H)$-set.

Theorem 2.8: For any two graphs G and H, $\gamma_{r2}(G \times H) \leq \gamma_{r2}(G)\gamma_{r2}(H)$ where $G \times H$ denote the cross product of G and H.

Proof: Let D be a $\gamma_{r2}(G)$-set and D' be a $\gamma_{r2}(H)$-set.

Let $(u,v) \in G \times H$. Then there exist at least two vertices, respectively, $a, b \in D$ and $a', b' \in D'$ such that *a*, *b* are adjacent to *u* and a', b' are adjacent to *v*. Thus $(a,a'), (b,b') \in D \times D'$ dominates $(u,v) \in G \times H$. Now let $x \in G \setminus D$ and $y \in H \setminus D'$. Then there exist at least one vertex, respectively, $c \in G \setminus D$ and $d \in H \setminus D'$ such that *c* is adjacent to *x* and *d* is adjacent to *y*. Thus $(x, y) \in (G \times H) \setminus (D \times D')$ is adjacent to $(c, d) \in (G \times H) \setminus (D \times D')$. Hence $D \times D'$ is a restrained 2-dominating set for $G \times H$. Since $|D \times D'| \le |D||D'|$, the theorem follows.

Theorem 2.9: If T is a tree, then $\gamma_{r2}(T) \ge 1 + \Delta(T)$. Moreover the equality holds if and only if T is a star.

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